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Exploring the Calkin-Wilf Tree: Subtrees and the Births of Numbers

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1 Introduction

In mathematics, we can describe sets as either finite or infinite. If a set is infinite, we can further describe this set as either countably or uncountably infinite. If there exists a bijection from a set to the set of natural numbers, then we say that set is countably infinite. The set of integers is called countably infinite since we can enumerate the integers in a way that allows us to create a mapping from the set of integers to the set of natural numbers, namely $f(x)=2x$ if $x>0$, and $f(x)=2|x|+1$ if $x\leq 0$. We say a set is uncountably infinite if a bijective mapping from that set to the set of natural numbers does not exist. For example, the set of real numbers is uncountably infinite since we cannot define a bijection from the set of real numbers to the set of natural numbers. Over a hundred years ago Georg Cantor created a one-to-one and onto mapping of the rational numbers to the set of natural numbers, hence proving that the rational numbers are countably infinite [8]. When we discuss set theory and the concept of the rational numbers being countably infinite, his proof is the one that we most often see. However, in 1999, Neil Calkin and Herbert Wilf gave an alternate proof of this fact using a fraction tree which became known as the Calkin-Wilf tree [7].

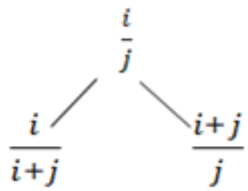

Figure 1: How fractions are formed in the Calkin-Wilf tree

The Calkin-Wilf tree is a binary tree whose vertices are labeled by fractions. The root of the tree is labeled by $\frac{1}{1}$. Each vertex with the label $\frac{i}{j}$ has two children: a left child, $\frac{i}{i+j}$, and a right child, $\frac{i+j}{j}$, see Figure 1. To generate the second level of the tree we create the right and left children as described above to get the fractions $\frac{1}{2}$ and $\frac{2}{1}$. We then take the children of these fractions to get the fractions $\frac{1}{3}, \frac{3}{2}, \frac{2}{3}$ and $\frac{3}{1}$. We continue this process infinitely, creating an infinite fraction tree. The first five levels of the Calkin-Wilf tree can be seen in Figure 2.

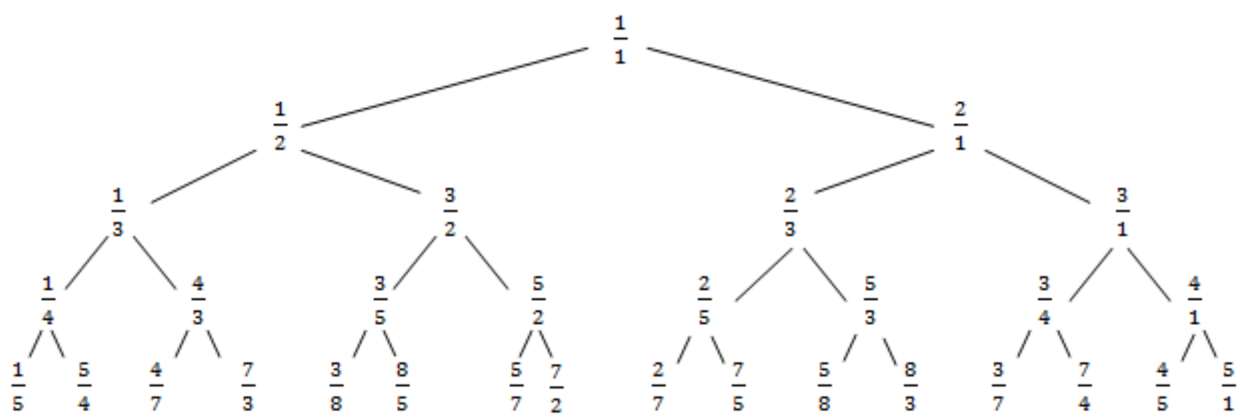

Figure 2: The first five levels of the Calkin-Wilf tree

There are several interesting properties of the fractions that appear in the Calkin-Wilf tree as shown in [7]. One interesting property is that the numerator and the denominator of each fraction are relatively prime. Moreover, every rational number occurs at some vertex once and only once. This last property is pivotal since it presents a way to enumerate the rational numbers so that every fraction is listed exactly once in the tree, creating a bijection to the natural numbers.

Several authors have explored the tree and its many interesting properties. Research has been done on the diagonals of the tree [9], the births of numbers within the tree [12], as well as the occurrence of Fibonacci numbers within the tree [4]. In the first part of this paper, we will discuss the births, or first occurrence, of numbers within the tree. We will then discuss the births of special sequences within the tree. In the last section we will discuss subtrees and determine which numbers do not occur in any given subtree.

2 Births

In this part of the paper we will examine where numbers first occur as either the numerator or denominator of a fraction in the tree. For example, looking at Figure 2, we see 3 for the first time in the denominator of the fraction in the 4^{th} position of the tree and we see 7 for the first time in the denominator of the fraction in the 18^{th} position of the tree. We define the birth of a natural number to be the first time that number occurs within the tree as either a numerator or a denominator. Hence the birth of 3 is in the 4^{th} position and the birth of 7 is in the 18^{th} position.

We will first consider the level each natural number n is born on. Upon looking at the first few levels of the tree (Figure 2) we notice that the birth of 1 is on the 1^{st} level, the birth of 2 is on the 2^{nd} level and the birth of 3 is on the 3^{rd} level. The numbers 4 and 5 are both born on the 4^{th} level. The births of 7 and 8 appear on the 5^{th} level, whereas 6 will have its birth later in the tree, on the 6^{th} level. The table below shows the birth level of the first 30 births.

Number	Birth Level	Number	Birth Level
1	1	16	7
2	2	17	7
3	3	18	7
4	4	19	7
5	4	20	8
6	6	21	7
7	5	22	8
8	5	23	8
9	6	24	8
10	6	25	8
11	6	26	8
12	6	27	8
13	6	28	8
14	7	29	8
15	7	30	8

From this table, it is easy to see that numbers that are numerically close to one another appear to have births on levels that are close together. Notice also that although the birth levels seem somewhat predictable, once in a while a birth won't behave as expected, as we see with 6 and 20. We can also see from this chart that higher levels of the tree contain more births than the previous levels. The graph below shows the birth level of the first 1,000 numbers.

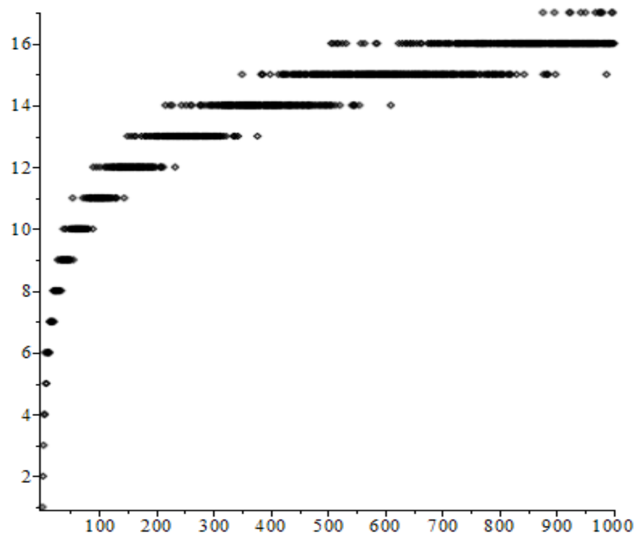

Figure 3: Birth level of the first 1000 births

Looking at this graph we notice that it takes the form of a logarithmic function, growing quickly at first and then leveling out. This graph suggests that the sequence of birth levels is logarithmic.

We can also observe these births in terms of their position within the tree. The following table shows the birth position of the first 30 births.

Number	Birth Position	Number	Birth Position
1	1	16	92
2	2	17	76
3	4	18	74
4	8	19	82
5	10	20	188
6	32	21	84
7	18	22	140
8	20	23	138
9	34	24	152
10	38	25	150
11	36	26	146
12	44	27	154
13	42	28	266
14	68	29	148
15	72	30	164

Looking at this table we notice that every birth occurs in an even position of the tree and consequently, a left child. This is due to the structure of the tree. Referring back to the tree definition given in Figure 1, the only way to obtain a new number is by adding i and j. Thus, if $i+j$ gives us a birth, then we will first encounter $i+j$ in the denominator of the left child (since we read the tree from left to right).

Similar to what we saw in the birth levels, numbers that are numerically close together tend to have their births relatively close to one another. However, the birth positions jump around quite a bit. For example,19 has its birth in the 82^{nd} position, but then the birth of 20 jumps to the 188^{th} position, 21 has its birth back down in the 84^{th} position, and 22's birth jumps back up to the 140^{th} position. The graph below shows the birth of the first 1000 natural numbers.

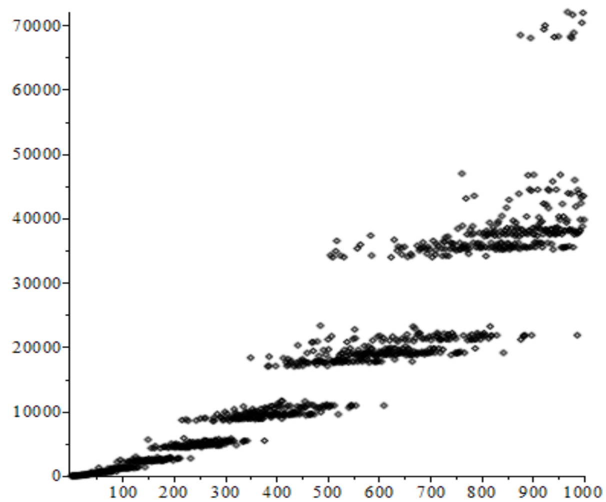

Figure 4: Birth position of the first 1000 natural numbers

This graph has the appearance of an exponential function, suggesting that the birth positions grow exponentially.

Examining the births further, we can give an upper bound for the birth of a number. We will do this by first proving that a number n will always occur in the left most fraction on the n^{th} level.

Proposition 1: $\frac{1}{n}$ is the left most fraction on the n^{th} level of the Calkin-Wilf tree for $n \geq 1$.

Proof: We will prove this by induction.

Base Case: Let $n = 1$. Since $\frac{1}{1}$ is the left most fraction on the 1^{st} level, the base case holds.

Inductive Step: Assume $\frac{1}{k}$ is the left most fraction on the k^{th} level. Then its left child will give us the left most fraction on the $k+1^{\text{st}}$ level. Since its left child is $\frac{1}{k+1}$, by the Principle of Mathematical Induction $\frac{1}{n}$ is the left most fraction of the n^{th} level for $n \geq 1$. ■

Since we have proved that n occurs in this position, we can give an upper bound for the birth position of the number n.

Corollary 1: The upper bound for the birth position of a natural number n is 2^{n-1}.

Proof: Notice that the left most child on the n^{th} level is in the 2^{n-1} position. Since we know n will occur in the denominator of the fraction in this position, if n has not already occurred earlier in the tree, then the latest we will find the birth for n is in the 2^{n-1} position. ■

This upper bound is met quite early for the birth of number 6. However, the birth of most *n,* particularly large *n,* will occur much earlier.

The Birth of Fibonacci Numbers

To further explore births, we considered the births of some special sequences. Consider the Fibonacci sequence defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, where $F_0=1$ and $F_1=1$. This recurrence relation gives us the following start to the sequence: 1, 1, 2, 3, 5, 8, 13, …

The Fibonacci births are highlighted in blue in the tree below. If we look at their position in the following portion of the tree we will notice that parents of fractions that contain Fibonacci numbers also contain Fibonacci numbers. For example, the fraction $\frac{3}{5}$ on the 3rd level contains the birth of 5. Notice that its parent is the fraction $\frac{3}{2}$, whose numerator and denominator are two previous Fibonacci numbers.

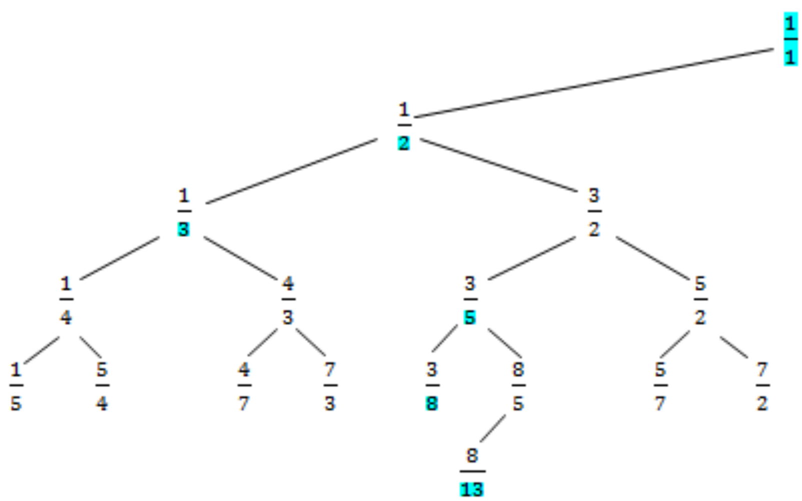

Figure 5: Left half of the Calkin-Wilf Tree with Fibonacci births

We can make a few observations about the Fibonacci numbers within the tree. First, we will make an observation about the Fibonacci ratios in the tree. A Fibonacci ratio is a fraction that contains two consecutive Fibonacci numbers in the form $\frac{F_n}{F_{n+1}}$ or $\frac{F_{n+1}}{F_n}$.

Proposition 2: Let $p\left(\frac{a}{b}\right)$ denote the position of the fraction $\frac{a}{b}$ within the Calkin-Wilf tree. Then for all $n \geq 1$,

$$p\left(\frac{F_n}{F_{n+1}}\right) = \begin{cases} \frac{5*2^n-2}{3}, & \text{if } n \text{ is even} \\ \frac{2^{n+2}-2}{3}, & \text{if } n \text{ is odd} \end{cases}$$

and

$$p(\frac{F_{n+1}}{F_n}) = \begin{cases} \frac{2^{n+2}-1}{3}, \text{if } n \text{ is even} \\ \frac{5*2^n-1}{3}, \text{if } n \text{ is odd.} \end{cases}$$

Proof: We will prove this by using induction.

Base Case: Let n=1

Then, $\frac{F_n}{F_{n+1}} = \frac{F_1}{F_2} = \frac{1}{2}$, $\frac{F_{n+1}}{F_n} = \frac{F_2}{F_1} = \frac{2}{1}$. Looking at the tree, $p\left(\frac{1}{2}\right) = 2$ and $p\left(\frac{2}{1}\right) = 3$. Since n is odd we find $p\left(\frac{1}{2}\right) = \frac{2^{n+2}-2}{3} = \frac{6}{3} = 2$ and $p\left(\frac{2}{1}\right) = \frac{5*2^1-1}{3} = \frac{9}{3} = 3$, respectively. Thus the base case holds.

We must also check the even case. Let n=2.

Then, $\frac{F_n}{F_{n+1}} = \frac{F_2}{F_3} = \frac{2}{3}$ and $\frac{F_{n+1}}{F_n} = \frac{F_3}{F_2} = \frac{3}{2}$. Looking at the tree, $p\left(\frac{2}{3}\right) = 6$ and $p\left(\frac{3}{2}\right) = 5$. Since n is even we find $p\left(\frac{F_2}{F_3}\right) = \frac{5*2^2-2}{3} = \frac{18}{3} = 6$ and $p\left(\frac{F_3}{F_2}\right) = \frac{2^{2+2}-1}{3} = \frac{15}{3} = 5$. Thus, the base case holds.

Inductive Step: Assume our property is true for n=k i.e.

$$p(\frac{F_k}{F_{k+1}}) = \begin{cases} \frac{5*2^k-2}{3}, \text{if } k \text{ is even} \\ \frac{2^{k+2}-2}{3}, \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad p(\frac{F_{k+1}}{F_k}) = \begin{cases} \frac{2^{k+2}-1}{3}, \text{if } k \text{ is even} \\ \frac{5*2^k-1}{3}, \text{if } k \text{ is odd.} \end{cases}$$

We want to show this property is true for n=k+1 i.e.

$$p(\frac{F_{k+1}}{F_{k+2}}) = \begin{cases} \frac{5*2^{k+1}-2}{3}, \text{if } k+1 \text{ is even} \\ \frac{2^{k+3}-2}{3}, \text{if } k+1 \text{ is odd} \end{cases} \quad \text{and} \quad p(\frac{F_{k+2}}{F_{k+1}}) = \begin{cases} \frac{2^{k+3}-1}{3}, \text{if } k+1 \text{ is even} \\ \frac{5*2^{k+1}-1}{3}, \text{if } k+1 \text{ is odd.} \end{cases}$$

Consider first the Fibonacci ratio $\frac{F_{k+2}}{F_{k+1}}$. Since its numerator is larger than its denominator, it must be a right child. The parent of this fraction would then be $\frac{F_{k+2}-F_{k+1}}{F_{k+1}} = \frac{F_k}{F_{k+1}}$, another Fibonacci ratio. The right child of the fraction in the n^{th} position is in the $2n+1^{\text{st}}$ position. So, to get the position of $\frac{F_{k+2}}{F_{k+1}}$, the right child of $\frac{F_k}{F_{k+1}}$, we take the position of $\frac{F_k}{F_{k+1}}$, multiply it by 2 and add 1. So

$$p(\frac{F_{k+2}}{F_{k+1}}) = \begin{cases} 2\left(\frac{5*2^k-2}{3}\right) + 1 = \frac{10*2^k-4+3}{3} = \frac{5*2^{k+1}-1}{3}, \text{if } k+1 \text{ is odd } (k \text{ is even}) \\ 2\left(\frac{2^{k+2}-2}{3}\right) + 1 = \frac{2*2^{k+2}-4+3}{3} = \frac{2^{k+3}-1}{3}, \text{ if } k+1 \text{ is even } (k \text{ is odd}). \end{cases}$$

Now consider the Fibonacci ratio $\frac{F_{k+1}}{F_{k+2}}$. Since its denominator is larger than its numerator, it must be a left child. The parent of this fraction would then be $\frac{F_{k+1}}{F_{k+2}-F_{k+1}} = \frac{F_{k+1}}{F_k}$. The left child of the fraction in the n^{th} position is in the $2n^{\text{th}}$ position. So, to get the position of $\frac{F_{k+1}}{F_{k+2}}$, the left child of $\frac{F_{k+1}}{F_k}$, we take the position of $\frac{F_{k+1}}{F_k}$ and multiply it by 2. So we have

$$p\left(\frac{F_{k+1}}{F_{k+2}}\right) = \begin{cases} 2\left(\frac{2^{k+2}-1}{3}\right) = \frac{2^{k+3}-2}{3}, \text{if } k+1 \text{ is odd } (k \text{ is even}) \\ 2\left(\frac{5*2^{k}-1}{3}\right) = \frac{5*2^{k+1}-2}{3}, \text{if } k+1 \text{ is even } (k \text{ is odd}). \end{cases}$$

Thus the statement is true when $n = k+1$. Hence by the Principle of Mathematical Induction,

$$p\left(\frac{F_n}{F_{n+1}}\right) = \begin{cases} \frac{5*2^n-2}{3}, \text{if } n \text{ is even} \\ \frac{2^{n+2}-2}{3}, \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad p\left(\frac{F_{n+1}}{F_n}\right) = \begin{cases} \frac{2^{n+2}-1}{3}, \text{if } n \text{ is even} \\ \frac{5*2^n-1}{3}, \text{if } n \text{ is odd} \end{cases}$$

for all $n \geq 1$. ■

We will use this proposition and a result from Calkin and Wilf to show what level a Fibonacci number is born on. Calkin and Wilf show in [6] that the maximum number we see as either a numerator or denominator for the first n fractions for for $n < 2^k$ is F_k. Because of this, we can prove what level a Fibonacci number is born on.

Theorem 1: The Fibonacci number F_k is born on the k^{th} level of the Calkin-Wilf tree.
Proof: By [6], the maximum number we see as either a numerator or denominator for the first 2^{k-1} fractions is F_{k-1}. Since 2^{k-1} is the position of the first fraction on the k^{th} level, this means the largest number we see as a numerator or denominator before the k^{th} level is F_{k-1}. Since $F_k > F_{k-1}$, we have that F_k does not occur before the k^{th} level of the tree. By Proposition 2, we know F_k occurs in the denominator of the fraction in the $\frac{2^{k+1}-2}{3}$ position if k is even, and in the numerator of the fraction in the $\frac{2^{k+1}-1}{3}$ position if k is odd. Since these positions are on the k^{th} level, we have that F_k is born on the k^{th} level. ■

Together, Proposition 2 and Theorem 1 work together to give us insight as to where the birth of F_k can occur. Recall that births only occur in the denominator of a fraction. Referring back to Proposition 2, if we let $k=n+1$, then F_k occurs as the denominator of the fraction in the position $\frac{2^{k+1}-2}{3}$, if k is even. Note that this position is on the k^{th} level. Moreover, F_k occurs as the numerator on the k^{th} level in the $\frac{2^{k+1}-1}{3}$ position if k is odd. But, since we are interested in where F_k occurs as the denominator, we take the fraction previous to this one. Hence F_k occurs in the denominator of the fraction in the $\frac{2^{k+1}-1}{3} - 1$ position if k is odd. Since we know F_k occurs in these positions on the k^{th} level and by Theorem 1 we know that F_k does not occur before this level, we make the following conjecture about the birth position of the Fibonacci numbers.

Conjecture 1: The birth position of a Fibonacci number F_k is $\frac{2^{k+1}-2}{3}$ if k is even and $\frac{2^{k+1}-1}{3} - 1$ if k is odd, for $k \geq 2$.

The Birth of Lucas Numbers

We now consider the Lucas sequence defined by the following recurrence relation: $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, where $L_0=2$ and $L_1=1$. This recurrence relation gives us the following sequence: 2, 1, 3, 4, 7, 11, 18, 29, ...

The Lucas births are highlighted in green in the tree below. Similar to the Fibonacci numbers, if we look at the position of the birth of Lucas numbers in the following portion of the tree we will notice that parents of fractions that contain Lucas numbers also contain Lucas numbers. For example, the fraction $\frac{4}{7}$ on the 5th level, contains the birth of 7. Notice that its parent is the fraction $\frac{4}{3}$, whose numerator and denominator are two previous Lucas numbers.

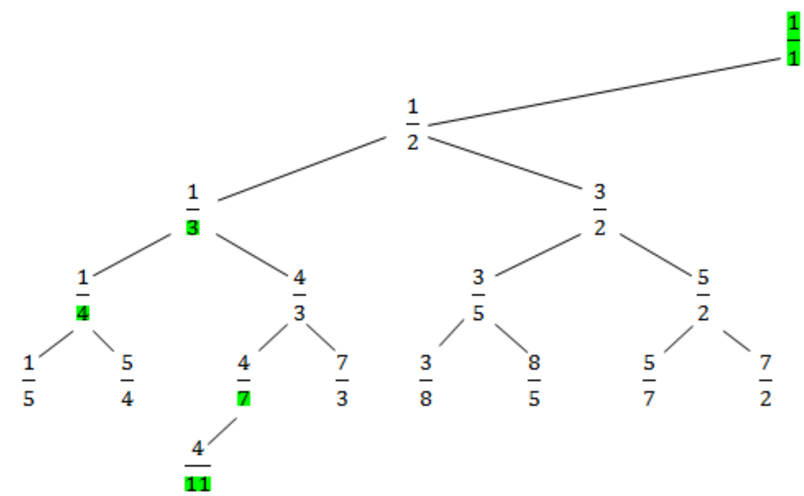

Figure 6: Left half of the Calkin-Wilf Tree with Lucas births

We can make a few observations about the Lucas numbers within the tree. First, we will make an observation about the Lucas ratios. A Lucas ratio is a fraction that contains two consecutive Lucas numbers in the form $\frac{L_n}{L_{n+1}}$ or $\frac{L_{n+1}}{L_n}$.

Proposition 3: Let $p\left(\frac{a}{b}\right)$ denote the position of the fraction $\frac{a}{b}$ within the Calkin-Wilf tree. Then for all $n \geq 2$,

$$p\left(\frac{L_n}{L_{n+1}}\right) = \begin{cases} 2^n + \frac{2(1-2^{n+2})}{-3}, \text{if } n \text{ is even} \\ 2^n + \frac{2(1-2^{n+1})}{-3}, \text{if } n \text{ is odd} \end{cases}$$

and

$$p\left(\frac{L_{n+1}}{L_n}\right) = \begin{cases} 2^n + \frac{1-2^{n+2}}{-3}, \text{if } n \text{ is even} \\ 2^n + \frac{1-2^{n+3}}{-3}, \text{if } n \text{ is odd.} \end{cases}$$

Proof: We will prove this by using induction.

Base Case: Let n=2.

Then, $\frac{L_n}{L_{n+1}} = \frac{L_2}{L_3} = \frac{3}{4}$ and $\frac{L_{n+1}}{L_n} = \frac{L_3}{L_2} = \frac{4}{3}$. Looking at the tree, $p\left(\frac{3}{4}\right) = 14$ and $p\left(\frac{4}{3}\right) = 9$. Since n is even we find $p\left(\frac{L_2}{L_3}\right) = 2^2 + \frac{2(1-2^{2+2})}{-3} = 4 + \frac{-30}{-3} = 14$ and $p\left(\frac{L_3}{L_2}\right) = 2^2 + \frac{1-2^{2+2}}{-3} = 4 + \frac{-15}{-3} = 9$ respectively. Since $\frac{4}{3}$ is in the 14th position of the tree and $\frac{3}{4}$ is in the 9th position in the tree, the base case holds.

We must also check the odd case. Let n=3.

Then, $\frac{L_n}{L_{n+1}} = \frac{L_3}{L_4} = \frac{4}{7}$ and $\frac{L_{n+1}}{L_n} = \frac{L_4}{L_3} = \frac{7}{4}$. Looking at the tree, $p\left(\frac{4}{7}\right) = 18$ and $p\left(\frac{7}{4}\right) = 29$. Since n is odd we find $p\left(\frac{L_3}{L_4}\right) = 2^3 + \frac{2(1-2^{3+1})}{-3} = 8 + \frac{-30}{-3} = 18$ and $p\left(\frac{L_4}{L_3}\right) = 2^3 + \frac{1-2^{3+3}}{-3} = 8 + \frac{-63}{-3} = 29$ respectively. Since $\frac{4}{7}$ is in the 18th position of the tree and $\frac{7}{4}$ is in the 29th position in the tree, the base case holds.

Induction: Assume our property is true for $n=k$ i.e.

$$p\left(\frac{L_k}{L_{k+1}}\right) = \begin{cases} 2^k + \frac{2(1-2^{k+2})}{-3}, \text{if } k \text{ is even} \\ 2^k + \frac{2(1-2^{k+1})}{-3}, \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad p\left(\frac{L_{k+1}}{L_k}\right) = \begin{cases} 2^k + \frac{1-2^{k+2}}{-3}, \text{if } k \text{ is even} \\ 2^k + \frac{1-2^{k+3}}{-3}, \text{if } k \text{ is odd} \end{cases}$$

We want to show this property is true for n=k+1 i.e.

$$p\left(\frac{L_{k+1}}{L_{k+2}}\right) = \begin{cases} 2^{k+1} + \frac{2(1-2^{k+3})}{-3}, \text{if } k+1 \text{ is even} \\ 2^{k+1} + \frac{2(1-2^{k+2})}{-3}, \text{if } k+1 \text{ is odd} \end{cases}$$

and

$$p\left(\frac{L_{k+2}}{L_{k+1}}\right) = \begin{cases} 2^{k+1} + \frac{1-2^{k+3}}{-3}, \text{if } k+1 \text{ is even} \\ 2^{k+1} + \frac{1-2^{k+4}}{-3}, \text{if } k+1 \text{ is odd.} \end{cases}$$

Consider first the children of the fraction $\frac{L_k}{L_{k+1}}$. Its left child is $\frac{L_k}{L_k+L_{k+1}} = \frac{L_k}{L_{k+2}}$ and its right child is $\frac{L_k+L_{k+1}}{L_{k+1}} = \frac{L_{k+2}}{L_{k+1}}$, a Lucas ratio. The right child of the fraction in the nth position is in the 2n+1st position. So, since the right child of $\frac{L_k}{L_{k+1}}$ is another Lucas ratio we take its position, multiply it by 2 and add 1 in order to get the position of $\frac{L_{k+2}}{L_{k+1}}$. So

$$p\left(\frac{L_{k+2}}{L_{k+1}}\right) = \begin{cases} 2\left(2^k + \frac{2(1-2^{k+2})}{-3}\right) + 1 = 2^{k+1} + \frac{4-2^{k+4}-3}{-3} = 2^{k+1} + \frac{1-2^{k+4}}{-3}, \text{if } k+1 \text{ is odd } (k \text{ is even}) \\ 2\left(2^k + \frac{2(1-2^{k+1})}{-3}\right) + 1 = 2^{k+1} + \frac{4-2^{k+3}-3}{-3} = 2^{k+1} + \frac{1-2^{k+3}}{-3}, \text{if } k+1 \text{ is even } (k \text{ is odd}). \end{cases}$$

Now consider the children of the fraction $\frac{L_{k+1}}{L_k}$. Its left child is $\frac{L_{k+1}}{L_k+L_{k+1}} = \frac{L_{k+1}}{L_{k+2}}$ and its right child is $\frac{L_k+L_{k+1}}{L_k} = \frac{L_{k+2}}{L_k}$. The left child of the fraction in the n^{th} position is in the $2n^{\text{th}}$ position. Since the left child of $\frac{L_{k+1}}{L_k}$ is another Lucas ratio we take its position and multiply it by 2 in order to get the position of $\frac{L_{k+1}}{L_{k+2}}$. So

$$p\left(\frac{L_{k+1}}{L_{k+2}}\right) = \begin{cases} 2\left(2^k + \frac{1-2^{k+2}}{-3}\right) = 2^{k+1} + \frac{2(1-2^{k+2})}{-3}, \text{if } k+1 \text{ is odd } (k \text{ is even}) \\ 2\left(2^k + \frac{1-2^{k+3}}{-3}\right) = 2^{k+1} + \frac{2(1-2^{k+3})}{-3}, \text{if } k+1 \text{ is even } (k \text{ is odd}). \end{cases}$$

Hence by the Principle of Mathematical Induction the position numbers of the Lucas ratios are:

$$p\left(\frac{L_n}{L_{n+1}}\right) = \begin{cases} 2^n + \frac{2(1-2^{n+2})}{-3}, \text{if } n \text{ is even} \\ 2^n + \frac{2(1-2^{n+1})}{-3}, \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad p\left(\frac{L_{n+1}}{L_n}\right) = \begin{cases} 2^n + \frac{1-2^{n+2}}{-3}, \text{if } n \text{ is even} \\ 2^n + \frac{1-2^{n+3}}{-3}, \text{if } n \text{ is odd} \end{cases}$$

for all $n \geq 2$. ■

Calkin and Wilf show in [6] that the maximum number we see as either a numerator or denominator for the first n fractions is F_k, for $n < 2^k$. We will first prove that $F_n < L_n < F_{n+1}$. Then, as a result of this, we can prove what level a Lucas number is born on.

Proposition 4: $F_n < L_n < F_{n+1}$ for $n \geq 2$

Proof: We will prove this by strong induction. Assume $F_k < L_k < F_{k+1}$ for all $2 < k < n$. Consider L_n. Then $L_n = L_{n-1} + L_{n-2} > F_{n-1} + F_{n-2} = F_n$. So $L_n > F_n$. Also $L_n = L_{n-1} + L_{n-2} < F_n + F_{n-1} = F_{n+1}$. So $F_n < L_n < F_{n+1}$. Hence, by the Principle of Mathematical Induction, $F_n < L_n < F_{n+1}$ for all $n \geq 2$. ■

Theorem 2: L_k is born on the k+1st level.

Proof: By [6], the maximum number we see as either a numerator or denominator for the first 2^k fractions is F_k. Since 2^k is the position of the first fraction on the k+1st level, this means the largest number we see as a numerator or denominator before the k+1st level is F_k. Since $L_k > F_k$ by Proposition 4, we have that L_k does not occur before the k+1st level of the tree. By Proposition 3 we know L_k occurs in the $2^k + \frac{2(1-2^{k+1})}{-3}$ position if k is odd, and in the $2^k + \frac{1-2^{k+2}}{-3}$ position if k is even. Since these positions are on the k+1st level, we have that L_k is born on the k+1st level. ■

Proposition 3 and Theorem 2 work together to give us some insight as to where the births of Lucas numbers occur. Since we know births only occur in the denominator of a fraction, if we refer back to Proposition 3, we note that on the k+1st level, if we let k=n+1 then L_k occurs as the denominator of the fraction in the position $2^{k-1} + \frac{2(1-2^k)}{-3}$, if k is even. Moreover, L_k occurs as the numerator on the k+1st level in the $2^{k-1} + \frac{1-2^{k+1}}{-3}$ position if k is odd. But, since we are interested in where L_k occurs as the denominator, we take the fraction previous to this one. Hence L_k occurs in the denominator of the fraction in the $2^{k-1} + \frac{1-2^{k+1}}{-3} - 1$ position if k is odd. Thus, we can make the following conjecture about the birth position of the Lucas numbers.

Conjecture 2: The birth position of the Lucas number L_k is $2^{k-1} + \frac{2(1-2^k)}{-3}$ if k is even and $2^{k-1} + \frac{1-2^{k+1}}{-3} - 1$ if k is odd.

3 Subtrees

In this section, we will investigate which numbers do not occur as either numerators or denominators in any given subtree. A subtree is simply a tree within the Calkin-Wilf tree that contains a vertex and all of its descendants. An example of a subtree can be seen in Figure 7. The root of this subtree is $\frac{5}{2}$, which is in the 11th position of the tree. Like the Calkin-Wilf tree, a subtree will go on infinitely. However, unlike the tree, a subtree will not contain every rational number. For example, $\frac{1}{2}$ will not appear in the subtree shown in Figure 7. We studied specific subtrees in order to determine which numbers do and do not occur as numerators or denominators.

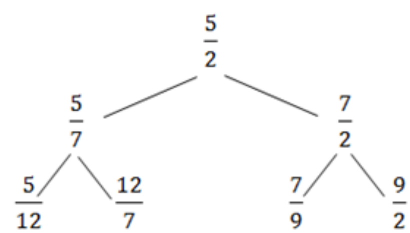

Figure 7: First three levels of the subtree rooted at $\frac{5}{2}$

Consider a general subtree with the root $\frac{a}{b}$, as seen in Figure 8. By definition of the Calkin-Wilf tree, its left child will be $\frac{a}{a+b}$, and its right child will be in the form $\frac{a+b}{b}$. Consider the fraction $\frac{a}{a+b}$, its left child will be $\frac{a}{a+(a+b)}$ or $\frac{a}{2a+b}$ and its right child will be $\frac{2a+b}{a+b}$. If we continue writing out more fractions, we will find that the numerator and denominator of each fraction are linear combinations of a and b. Therefore, if a natural number n is in the subtree rooted at $\frac{a}{b}$, then n can be written as a linear combination of a and b, namely $n = aX + bY$, where X and Y are natural numbers.

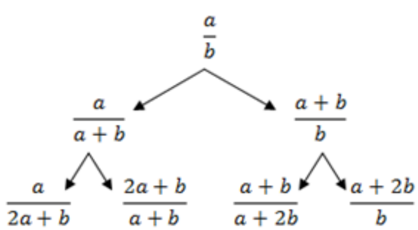

Figure 8: First three levels of the subtree rooted at $\frac{a}{b}$

We can see that a and b will always be in the subtree rooted at $\frac{a}{b}$. In this section, we will consider other numbers that occur in this subtree, in particular the numbers $n = aX + bY$, where $X, Y > 0$. Since $n=aX + bY$ is a linear Diophantine equation, we know it has integer solutions for X and Y when gcd(a,b)=1. We know that a and b are always relatively prime, so we know this equation will always have integer solutions. However, since we restrict X and Y to be natural numbers, this equation may not always have a solution satisfying this restriction.

If n cannot be written in the form $aX+bY$, where X and Y are natural numbers, then it will not appear in the subtree. We are interested in determining what numbers n will not appear as a numerator or denominator in the subtree rooted at $\frac{a}{b}$. In the remainder of this section, we will prove through a series of theorems which numbers do not occur in a subtree rooted at $\frac{a}{b}$ by considering numbers less than the sum of a and b, multiples of a and b, and numbers in the form $aX+bY$ where $X=Y$.

Consider the equation $aX + bY = c$. Since we restrict X and Y to be natural numbers, X and Y must be at least 1, except in the case where $c = a$ or b. Hence, the smallest c can be (other than a or b) is $a+b$. So, numbers less than the sum of a and b, aside from a and b themselves, will not occur in the subtree.

Consider the subtree rooted at $\frac{5}{2}$, as shown in Figure 7. We know that the numbers 1, 3, 4, and 6 will not occur in this subtree, since these numbers are less than 7, the sum of a and b. Moreover, any number that can be written in the form $5X+2Y$, where X and Y are distinct natural numbers, will appear in this subtree. Numbers that cannot be written in this form under these conditions will not appear in the tree.

There are numbers in addition to 1, 3, 4, and 6 that will not appear in the subtree. When considering the Linear Diophantine Equation $aX+bY$, we know that the Frobenius number, $ab-a-b$, is the largest number that has no non-negative integer solutions for X and Y. However, since we are only interested in the case where $X,Y>0$, there are numbers larger than this number that will not appear in this subtree as we've already seen in the $\frac{5}{2}$ subtree. It turns out that other than $a+b$, there will be no other $aX+bY$ where $X=Y$. This is due to the formation of fractions within a subtree. Since we obtain new fractions by adding multiples of the previous numerators and denominators, numerators and denominators in the left half of the subtree will have more a's than b's (meaning $X>Y$). Similarly, numerators and denominators in the right half of the subtree will always contain more b's than a's ($Y>X$). Hence $aX+bY$ does not appear as a linear combination in the subtree when $X=Y$, unless $X=Y=1$. We show this in the following theorem.

<u>**Theorem 3:**</u> The linear combination an_0+bn_0, $n_0 > 1$, does not occur in the subtree rooted at $\frac{a}{b}$.

Proof: Assume by way of contradiction that the linear combination an_0+bn_0 does occur in the subtree rooted at $\frac{a}{b}$. Since an_0+bn_0 is found in this subtree, it will be in one of its children's subtrees. The left child of $\frac{a}{b}$ will be $\frac{a}{a+b}$, and its right child will be $\frac{a+b}{b}$. If we consider the left child as the root of a new subtree, every number in that subtree can be written as a linear combination of its numerator and denominator, namely $aX+(a+b)Y$, where X and Y are positive integers. Suppose first that an_0+bn_0 will occur in this subtree. Then $an_0+bn_0 = aX+(a+b)Y$, for some X and Y. But $aX+(a+b)Y = a(X+Y)+b(Y)$, thus $an_0+bn_0 = a(X+Y)+b(Y)$. So, $n_0=X+Y=Y$. In order for $X+Y=Y$ to be true, $X=0$, a contradiction. Similarly, if we consider the right child we have that every number in that subtree can be written as $(a+b)X+bY$. Suppose that an_0+bn_0 will occur in this subtree. Then $an_0+bn_0 = (a+b)X+bY$, for some X and Y. But $(a+b)X+bY =$

$a(X)+b(X+Y)$, thus $an_0+bn_0 = a(X)+b(X+Y)$. So, $n_0=X=X+Y$, and $Y=0$, a contradiction. Hence, the linear combination an_0+bn_0 will not be in the subtree rooted at $\frac{a}{b}$ for $n_0>1$. ■

However, if there is an alternate way to represent the linear combination an_0+bn_0 as $aX+bY$ where X and Y are distinct integers, then we will find that number in the subtree. Consider the subtree rooted at $\frac{3}{2}$. By the previous result we have that the linear combination 3(4)+2(4) will not occur in this subtree. However, we can write 20=3(4)+2(4) as a linear combination of 3 and 4 where $X\neq Y$, namely 3(2)+2(7). Hence, 20 is in the $\frac{3}{2}$ subtree. The following theorem shows that there is an alternate way to express an_0+bn_0 if n_0 is greater than the minimum of a and b.

Theorem 4: Let $k_0 = an_0+bn_0$, where $n_0>1$. Then k_0 appears in the subtree rooted at $\frac{a}{b}$ iff $n_0 > \min\{a,b\}$.

Proof:

→ We will prove this by contrapositive, i.e. we will assume $n_0 \leq \min\{a,b\}$ and we will show that if $k_0 = an_0+bn_0$ then k_0 does not appear in the subtree rooted at $\frac{a}{b}$. Let $n_0 \leq \min\{a,b\}$ and let $k_0 = an_0+bn_0$. Assume without loss of generality that $a<b$, and so $n_0\leq a$. Consider the equation $k_0 = aX+bY$. We know that one solution to this equation is where $X=Y=n_0$. Then the general solution set for the equation $aX+bY$ is $S=\{(n_0-kb,\ n_0+ka) \mid k \text{ an integer}\}$. So $X=n_0-kb$ and $Y=n_0+ka$.

Case 1: Let $k<0$. Then $Y=n_0+ka$ would either be negative or 0, since we subtracting a multiple of a from n_0, a number that is less than or equal to a. Since Y is negative or 0, we have that k_0 does not occur in the subtree rooted at $\frac{a}{b}$.

Case 2: Let $k>0$. Then $X=n_0-kb$ would be negative, since we are subtracting a multiple of b from from n_0, which is less than b. Since X is negative, we have that k_0 does not occur in the subtree rooted at $\frac{a}{b}$.

Case 3: Let k=0, then $X=Y$ and by Theorem 3 we know that an_0+bn_0 cannot occur in the subtree rooted at $\frac{a}{b}$.

So in all three cases we have that have k_0 does not occur in the subtree rooted at $\frac{a}{b}$. Hence if $k_0 = an_0+bn_0$ and k_0 appears in the subtree rooted at $\frac{a}{b}$ then $n_0 > \min\{a,b\}$.

← Assume $n_0 > \min\{a,b\}$ and suppose without loss of generality that $\min\{a,b\} = a$, so $n_0>a$. Let $n_0 = a+m$, where $m>0$. Then,

$$\begin{aligned} k_0 &= an_0+bn_0 \\ &= a(a+m)+b(a+m) \\ &= a^2+am+ab+bm \\ &= a(a+m+b)+b(m). \end{aligned}$$

Note that $a+m+b \neq m$, since $a, b > 0$. Thus we have an alternate way to write $k_0 = an_0+bn_0$ as $aX+bY$ where $X \neq Y$, so k_0 will occur in the tree. Hence, if $n_0 > \min\{a,b\}$ then $k_0 = an_0+bn_0$ will occur in the tree. ■

Thus if we consider our $\frac{5}{2}$ subtree, since $2=\min\{2,5\}$, we know $5(2)+2(2)=14$ will not occur in this subtree, since there is no alternate way to write 14 as a linear combination of 5 and 2 where $X \neq Y$. However, since $3>\min\{2, 5\}$ we have that $5(3)+2(3)=21$ will occur in this subtree since there exists an alternate way to write 21 as a linear combination of 5 and 2, namely $5(1)+2(8)$. Hence in the $\frac{5}{2}$ subtree the following integers will not occur: 1, 3, 4, 6, and 14.

Since we can only add multiples of a and b to get new numerators and denominators in the subtree of $\frac{a}{b}$, we will never have a numerator or denominator expressed as as $aX+b(0)$ or $a(0)+bY$, unless X or Y is 1, giving us either a or b. So, if multiples of a and b are in subtree then we must be able to rewrite them as a linear combination of a and b where $X, Y >0$. The following three theorems will define which multiples of a and b will occur in the subtree rooted at $\frac{a}{b}$.

<u>Theorem 5</u>: Let $k_0 = cm$ where $c = a$ or b and $m>1$. If k_0 is in the subtree rooted at $\frac{a}{b}$ then $cm>ab$.

Proof:

Without loss of generality let $c=a$. Since k_0 is in the subtree rooted at $\frac{a}{b}$ then $k_0 = aX+bY$ for some natural numbers X and Y such that $X \neq Y$. Thus $am=aX+bY$. Since $a|am$, $a|(aX+bY)$. Since $a|(aX+bY)$ and $a|aX$, $a|bY$. Since a and b are relatively prime by a property of the Calkin-Wilf tree, then $a|Y$ by Euclid's Lemma. Thus since $Y > 0$, $Y=al$, for some natural number l. Then,

$$\begin{aligned} am &= aX + bY \\ &= aX + b(al) \\ &= ab(l) + aX > ab, \end{aligned}$$

since $a, b, l,$ and X are all natural numbers. Hence, if $k_0 = cm$ is in the subtree rooted at $\frac{a}{b}$ then $cm>ab$. ■

This theorem tells us that if a multiple of a or b is in the subtree, then that multiple must be greater than ab. The next theorem gives us a way to determine which multiples of the minimum of a and b will occur in the tree.

<u>Theorem 6:</u> Let $c= \min\{a,b\}$ and let $k_0=cm$, where m is a natural number. Then k_0 is in the subtree rooted at $\frac{a}{b}$ iff $cm>ab$ and $cm \neq c(a+b)$.

Proof:

→ Since $k_0=cm$ is in the subtree rooted at $\frac{a}{b}$ then by Theorem 5, $cm>ab$. By Theorem 4, since $c=\min\{a, b\}$, $c(a+b)$ will not appear in the subtree. Thus $cm \neq c(a+b)$.

← Assume $cm>ab$ and $cm \neq c(a+b)$. Without loss of generality let $c=a$. Then, $am=cm>ab$. Hence, $m>b$ *and thus* $m=b+l$ for some natural number l. Then,

$$\begin{aligned} am &= a(b+l) \\ &= ab+al \\ &= a(l)+b(a). \end{aligned}$$

Since $cm=am$, we have $am \neq a(a+b)$. Also $am=a(l)+b(a)$, so $a(l)+b(a) \neq a(a+b)$, hence $l \neq a$. Since there exists natural numbers X and Y (namely, l and a) such that $X \neq Y$ where $cm=aX+bY$. Hence, $k_0=cm$ is in the subtree rooted at $\frac{a}{b}$.

Thus, $k_0=cm$ is in the subtree rooted at $\frac{a}{b}$ iff $cm \geq ab$ and $cm \neq c(a+b)$. ■

Applying this theorem to the subtree rooted at $\frac{5}{2}$, we can conclude that multiples of 2 will appear in this subtree as long as the multiple is greater than 10 and not equal to 14. Thus the numbers 4, 6, 8, 10, and 14 do not occur as a numerator or denominator in this subtree in addition to 1 and 3 that we have seen before.

We will now consider multiples of the maximum of the numerator and denominator. We can prove that multiples of the maximum of the numerator and denominator (say dm) will occur in the subtree iff $dm>ab$.

<u>Theorem 6</u>: Let $d=\max\{a, b\}$ and let $k_0=dm$, where m is a natural number. Then k_0 is in the subtree rooted at $\frac{a}{b}$ iff $dm>ab$.

Proof:

→ Since $k_0=dm$ is in the subtree rooted at $\frac{a}{b}$ then by Theorem 4 $dm>ab$.

← Without loss of generality let $d=a$. Assume $dm>ab$. Then $am=dm>ab$. Hence $m>b$, so $m=b+l$ for some natural number l. Then,

$$\begin{aligned} am &= a(b+l) \\ &= a(l)+b(a). \end{aligned}$$

If $l \neq a$ then there exists natural numbers X and Y such that $X \neq Y$ where $dm=aX+bY$. If $a=l$ then since we let a be the maximum of a and b, then by Theorem 3, there exists an alternate way to write $am=aX+bY$ where $X \neq Y$. Hence, $k_0=dm$ is in the subtree rooted at $\frac{a}{b}$.

Hence, $k_0=dm$ is in the subtree rooted at $\frac{a}{b}$ iff $dm>ab$. ■

As a result of this theorem, multiples of 5 will occur in in the subtree rooted at $\frac{5}{2}$ if they are greater than 10. Since all multiples of 5, other than 5 and 10, are greater than 10, no other numbers satisfy this condition. Thus the only numbers that will not appear in the subtree rooted at $\frac{5}{2}$ are 1, 3, 4, 6, 8, 10, and 14.

With these results, we can enumerate the integers that do not appear as the numerator or denominator in any fraction in a given subtree.

Theorem 7: In the subtree rooted at $\frac{a}{b}$ the following numbers will not occur:

- $k_0 < a+b$, where $k_0 \neq a$ or b
- $k_0 = an_0+bn_0$, $n_0 \leq \min\{a,b\}$
- $k_0 = am \leq ab$
- $k_0 = bm \leq ab$.

Hence, if a number k_0 does not satisfy these conditions then we will be able to find k_0 in some fraction in the subtree rooted at $\frac{a}{b}$.

In this last section we have described all numbers that do not occur in the subtree rooted at $\frac{a}{b}$ and thus we know which numbers do occur as a numerator or denominator in a fraction within the subtree. We have also discussed some properties of births as well as made conjectures for where the birth of some special sequences occur. In the future we hope that the knowledge of where numbers do not occur in the Calkin-Wilf tree will help us to determine the birth of any number within the tree.

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